$\mathbb{C}^*$-ACTIONS ON GRASSMANN BUNDLES AND THE CYCLE AT INFINITY

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SİNAN SERTÖZ

Abstract. We describe the Grassmann Graph construction of MacPherson in the analytic category using a $\mathbb{C}^*$-action and the corresponding Bialynicki-Birula decomposition. It is shown that the cycle at infinity is analytic in the compact Kaehler case.

0. Introduction.

This paper describes the Grassmann Graph construction of MacPherson in the analytic category using $\mathbb{C}^*$-actions. The details of the algebraic case can be found in [1].

In section 1 we summarize the decomposition theorem of Bialynicki-Birula in the compact Kaehler case, [2], [3]. Section 2 describes a $\mathbb{C}^*$-action on Grassmann manifolds and gives the corresponding Bialynicki-Birula decomposition. Examples are given in the next section. In section 4 this $\mathbb{C}^*$ action is carried on to Grassmann bundles and $\mathbf{Z}_\infty$, the cycle at infinity corresponding to a bundle morphism is defined. It is shown that in the compact Kaehler case $\mathbf{Z}_\infty$is an analytic cycle. The graph construction is finally accomplished in section 5. Examples are given in section 6.

Verdier uses the existence of a closed analytic space S which contains the closure of the graph in transcribing for analytic spaces the results of MacPherson, [9, section 5, proposition], [6]. We show in theorem 1 that in the compact Kaehler case S not only contains but is equal to the closure of the graph.

1. Bialynicki-Birula decomposition

The references for this section are [2] for the algebraic case and [3] for the complex case. There is also a clear summary in [4, section Ic].

Let M be a compact Kaehler manifold with a $\mathbb{C}^*$-action on it. Let this $\mathbb{C}^*$-action have nontrivial fixed point set B with components $B_1, \ldots, B_m$. The components of the fixed point set are complex submanifolds of M. For $\lambda \in \mathbb{C}^*$ and $p \in M$ let $\lambda \cdot p$ denote the action of λ on p. The $\mathbb{C}^*$-action extends to a meromorphic map

$$\mathbb{P}^1 \times \{p\} \longrightarrow M$$

hence $\lim_{\lambda\to 0} \lambda \cdot p$ and $\lim_{\lambda\to\infty} \lambda \cdot p$ exist in M. Clearly these limits are in B. There are two canonical decompositions of M into invariant complex submanifolds.

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Define

$$M_i^+ = \{p \in M | \lim_{\lambda \to 0} \lambda \cdot p \in B_i\}$$

for $i = 1, ..., m$. Each M_i^+ is a complex submanifold of M and

$$M = \bigcup M_i^+, \;\; 1 \leq i \leq m.$$

This is called the *plus decomposition* of M. Similarly the *minus decomposition* is defined as

$$M_i^- = \{p \in M | \lim_{\lambda \to \infty} \lambda \cdot p \in B_i\}$$

for $i = 1, ..., m$. Each M_i^- is a complex submanifold and similarly

$$M = \bigcup M_i^-, \;\; 1 \leq i \leq m.$$

There are two distinguished components of the fixed point set B, say B_1 and B_m, which are determined by the property that M_1^+ and M_m^- are open and dense in M. B_1 is called the *source* and B_m is called the *sink*.

2. $\mathbb{C}^*$-action on $G(k,n)$.

In this section we describe a particular $\mathbb{C}^*$-action on $G(k,n)$, the Grassmann manifold of k-planes in n-space. Fix a coordinate system on $\mathbb{C}^n$. We will use the representation of $G(k,n)$ by matrices. Any point $p \in G(k,n)$ can be represented by a $k \times n$-matrix A of rank k. Two such matrices A and B represent the same point in $G(k,n)$ if there is an invertible $k \times k$-matrix $g \in G(k, \mathbb{C})$ such that $gA = B$. For a $k \times n$-matrix A of rank k set $[A]$ = the row space of A.

Given a $k \times n$-matrix $A = (a_{ij}),\ 1 \leq i \leq k,\ 1 \leq j \leq n$ define two submatrices

$$A_1 = (a_{ij}, \;\; 1 \leq i, j \leq k$$

and

$$A_2 = (a_{ij}, \; 1 \leq i \leq k, \;\; k+1 \leq j \leq n.$$

A_1 is a $k \times k$-matrix and A_2 is a $k \times (n-k)$-matrix and $A = (A_1, A_2)$ is a partition of A.

Define a $\mathbb{C}^*$-action on $G(k,n)$

$$\mathbb{C}^* \times G(k,n) \longrightarrow G(k,n)$$

by

$$\lambda \cdot [A] = [(A_1, \lambda A_2)].$$

To describe the behaviour of this action define a subset X_{ij} of $G(k,n)$ as the set of all p in $G(k,n)$ which can be represented by a $k \times n$-matrix $A = (A_1, A_2)$ such that rank$A_1 = i$ and rank$A_2 = j$, where $k - \min\{k, n-k\} \leq i \leq k$ and $0 \leq j \leq \min\{k, n-k\}$. Let $B = (B_1, B_2)$ be another $k \times n$-matrix representing p. Then there is an invertible $k \times k$-matrix g such that $gA = B$.

$$gA_1 = B_1 \quad \text{and} \quad gA_2 = B_2.$$

Hence rankB_1 =rank(gA_1) =rank$A_1 = i$ and similarly rank$B_2 = j$, and the following definition of X_{ij} is well defines:

$$X_{ij} = \left\{ [A] \in G(k,n) \;\; \middle| \; \text{rank} A_1 = i, \;\; \text{rank} A_2 = j \; \right\}$$

where $k - \min\{k, n-k\} \leq i \leq k$ and $0 \leq j \leq \min\{k, n-k\}$. Necessarily we have $i + j \geq k$; to see this, recall that A represents a point in $G(k, n)$ hence has rank k, and if A_1 has rank i then A_2 must supply at least the remaining $k - i$ ranks.

To describe the behaviour of the $\mathbb{C}^*$-action that is defined above we prove the following lemmas.

Lemma 1. *$X_{i\,k-i}$ are the fixed point components of the $\mathbb{C}^*$-action, $k - \min\{k, n-k\} \leq i \leq k$.*

Proof: Let $[A] \in X_{i\,k-i}$, $A = (A_1, A_2)$. We first show that $\lambda \cdot [A] = [A]$. If $i = 0$, then $A_1 = 0$, and if $i = k$, then $A_2 = 0$. In both cases $\lambda \cdot [A] = [A]$. Assume $0 < i < k$. Then there exists an invertible $k \times k$-matrix g such that gA is of the form

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

where $B_1 \in GL(i\mathbb{C})$ and $B_2 \in GL(k-i, \mathbb{C})$. For $\lambda \in \mathbb{C}^*$ define h_λ to be the diagonal matrix $[1, \ldots, 1, 1/\lambda, \ldots, 1/\lambda]$, where the number of $1/\lambda$'s is $k - i$. We then have the following sequences of equalities:

$$\begin{aligned}
\lambda \cdot [A] &= \lambda \cdot [gA] \\
&= \lambda \cdot \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} B_1 & 0 \\ 0 & \lambda B_2 \end{pmatrix}\right] \\
&= \left[h_\lambda \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\right] \\
&= [A]
\end{aligned}$$

Thus we have proven that X_{ik-i} is a subset of the fixed point set. That in fact there are no other fixed points than $\cup X_{ik-i}$, $k - \min\{k, n-k\} \leq i \leq k$ follows from the results of the following two lemmas.

Lemma 2. *If $[A] \in X_{ij}$, then $\lim_{\lambda \to 0} \lambda \cdot [A] \in X_{ik-i}$, where*

$$k - \min\{k, n-k\} \leq i \leq k, \quad 0 \leq j \leq \min\{k, n-k\} \quad i + j \geq k.$$

In particular X_{k0} is the source.

Proof: If $i = 0$ or $i = k$, then X_{ij} is a component of the fixed point set as in Lemma 1. Assume $0 < i < k$. then there exists $g \in GL(k, \mathbb{C})$ such that

$$gA = \begin{pmatrix} & \vdots & 0 \\ B_1 & \vdots & \\ & \vdots & B_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & B_3 \end{pmatrix}$$

where $B_1 \in GL(i,\mathbb{C})$, $B_3 \in GL(k-i,\mathbb{C})$ and B_2 is a $(i+j-k)\times(n-k)$-matrix. Let h_λ be as in Lemma 1. then

$$h_\lambda \lambda g A = \begin{pmatrix} & \vdots & 0 \\ B_1 & \vdots & \\ & \vdots & \lambda B_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & B_3 \end{pmatrix}$$

and since $\lim_{\lambda\to 0} \lambda B_2 = 0$ we have

$$\begin{aligned} \lim_{\lambda\to 0} \lambda \cdot [A] &= \lim_{\lambda\to 0} [h_\lambda \lambda g A] \\ &= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3 \end{pmatrix}\right]. \end{aligned}$$

This last matrix is clearly in X_{ik-i} as claimed.

LEMMA 3. *If* $[A] \in X_{ij}$*, then* $\lim_{\lambda\to\infty} \lambda \cdot [A] \in X_{k-jj}$*, where*

$$k - \min\{k, n-k\} \le i \le k, \quad 0 \le j \le \min\{k, n-k\}.$$

In Particular $_{k-mm}$ *is the sink, where* $m = \min\{k, n-k\}$.

PROOF: If $i = 0$ or $i = k$, then X_{ij} is a fixed point component. Assume $0 < i < k$. there exists $g \in GL(k,\mathbb{C})$ such that

$$gA = \begin{pmatrix} B_1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ B_2 & \vdots & \\ & \vdots & B_3 \\ 0 & \vdots & \end{pmatrix}$$

where $B_1 \in GL(k-j,\mathbb{C})$, $B_3 \in GL(j,\mathbb{C})$ and B_2 is a $(i+j-k)\times k$-matrix. Then

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda \cdot [A] &= \lim_{\lambda\to\infty} [\lambda h_\lambda g A] \\ &= \lim_{\lambda\to\infty} \left[\begin{pmatrix} B_1 & 0 \\ \lambda^{-1} B_2 & \\ & B_3 \\ 0 & \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3 \end{pmatrix}\right]. \end{aligned}$$

This last matrix is in X_{k-jj} as desired.

These last two lemmas show that X_{ik-i} for $k - \min\{k, n-k\} \le i \le k$ are the only fixed point components and thus complete the proof of lemma 1.

We can apply these lemmas to examine the behaviour of Schubert cells under the action of $\mathbb{C}^*$ on the Grassmann manifold. we will adopt the terminology of Griffiths and Harris on Schubert cells. For details refer to [5, pp. 195-196].

Let $\{e_1, ..., V_n\}$ be the standard basis for $\mathbb{C}^n$ and $V_i = \text{span}\{e_1, ..., e_i\}$. Then $\{V_1, ..., V_n\}$ defines a flag. For any nonincreasing sequence of nonnegative integers between 0 and $n-k$ define a cell

$$W_a = \left\{[\Lambda] \in G(k,n) \;\middle|\; \dim(\Lambda \cap V_{n-k+i-a_i}) = i \right\}.$$

The sequence of nonincreasing integers $a = (a_1, ..., a_k)$ with $0 \leq a_i \leq n-k$ is called a Schubert symbol. For $[\Lambda] \in G(k,n)$, let A be a $k \times n$-matrixsuch that $[A] = [\Lambda]$. If $[A] \in W_a$ for some Schubert symbol $a = (a_1, ..., a_k)$, then the rank of the first $k \times (n-k+i-a_i)$ minor is i and the rank of the last $k \times (k-i+a_i)$ minor is $k-i$. The closure of W_a

$$\overline{W_a} = \left\{[\Lambda] \in G(k,n) \;\middle|\; \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \right\}$$

is called a Schubert variety. If A is a matrix representing $[\Lambda]$ as above, then $[\Lambda]$ is in $\overline{W_a}$ iff the rank of the first $k \times (n-k+i-a_i)$ minor of A is at most $k-i$. It is well known that $\overline{W_a}$is an analytic subvariety of $G(k,n)$ and the homology class of $\overline{W_a}$, denoted by σ_a, is independent of the flag used in its definition, [5, p. 196]. σ_a is called the Schubert cycle corresponding to $a = (a_1, ..., a_k)$. Regarding the behaviour of Schubert cycles under the $\mathbb{C}^*$-actionwe give the following corollary to the above lemmas:

Corollary 1. *All Schubert cycles of positive codimension in $G(k, 2k)$ lie in X_{ij}'s where $j < k$. In particular they do not flow to the sink, i.e. if $p \in \overline{W_a}$ then $\lim_{\lambda \to \infty} \lambda \cdot p$ is not in the sink.*

Proof: The codimension of $\overline{W_a}$for $a = (a_1, ..., a_k)$ is $\sum a_i$, [5, p. 196]. It suffices to prove the corollary for $a = (1, 0, ..., 0)$. For $[\Lambda] \in W_a$ let $A = (A_1, A_2)$ be a matrix representation where A is a $k \times n$-matrixof rank k, and A_1, A_2 are $k \times k$-matrices. The rank of the last $k \times k$ minor of A is of rank at most $k-1$. Hence in particular the rank of A_2 is not k, therefore $[A]$ is not in X_{ik}. Since the only points that flow to the sink belong to the components of the form X_{ik}, $[\Lambda]$ does not flow to the sink. In general if $a = (a_1, ..., a_k)$ with $a_1 \geq 1$ then the last $k \times (k + a_1 - 1)$ minor has rank at most $k-1$. Since $k + a_1 - 1 \geq k$, the rank of A_2 cannot be k. Hence $\overline{W_a}$does not flow to the sink. If $a_1 = 0$, then $a = (0, ..., 0)$ and $\overline{W_a}$does not have positive codimension.

Using the same notation as in the previous corollary we can generalize as follows:

Corollary 2. *Let $\overline{W_a}$, $a = (a_1, ..., a_k)$, be a Schubert variety in $G(k,n)$, where $a_1 \geq n - 2k + 1$. Then $\overline{W_a}$does not flow to the sink if $n \geq 2k$.*

Proof: Let $A = (A_1, A_2)$ be a $k \times n$-matrixwith rank k representing a point $[A]$ in $\overline{W_a}$. A_1 is a $k \times (n-k)$-matrix and $[A]$ will flow to the sink if rank A_2 is maximal. Since $n \geq 2k$ means $n - k \geq k$, the maximal rank of A_2 is k. the rank of the last $k \times (k + a_1 - 1)$ minor of A is at most $k-1$. By assumption $k + a_1 - 1 \geq n - k$, therefore the rank of A_2 cannot be k. Hence $\overline{W_a}$does not flow to the sink.

3. Examples

In examples 1 and 2 we assume that the $\mathbb{C}^*$-actionof the previous section is defined on the spaces $G(2,4)$ and $G(4,9)$.

1) $\mathbf{G(2,4)}$. In $G(2,4)$ we have defined the following sets:

$$X_{20}, X_{11}, X_{02}, X_{22}, X_{12}, X_{21}.$$

The first three sets are the fixed point sets. As $\lambda \to 0$ the elements of X_{21} and X_{22} flow to the source X_{20}, and the elements of X_{12} flow to X_{11}. As $\lambda \to \infty$ the elements of X_{22} and X_{12} flow to the sink X_{02}, and the elements of X_{21} flow to X_{11}.

See Figure 1 for the direction of these flows for each X_{ij} as $\lambda \to \infty$.

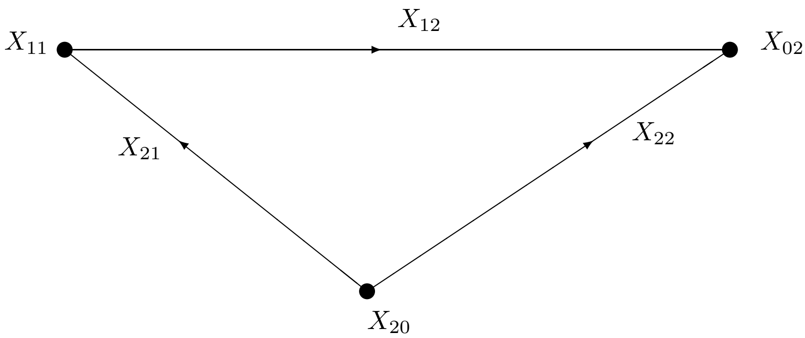

Fig. 1

2) $\mathbf{G(4,9)}$. For the direction of flow as $\lambda \to \infty$ see Figure 2. From the decomposition of $G(4,9)$ into X_{ij} it can be seen that the points that lie in

$$X_{13} \cup X_{33} \cap X_{31} \cap X_{32} \cap X_{22} \cap X_{23}$$

do not flow to the sink or the source under the action of $\mathbb{C}^*$.

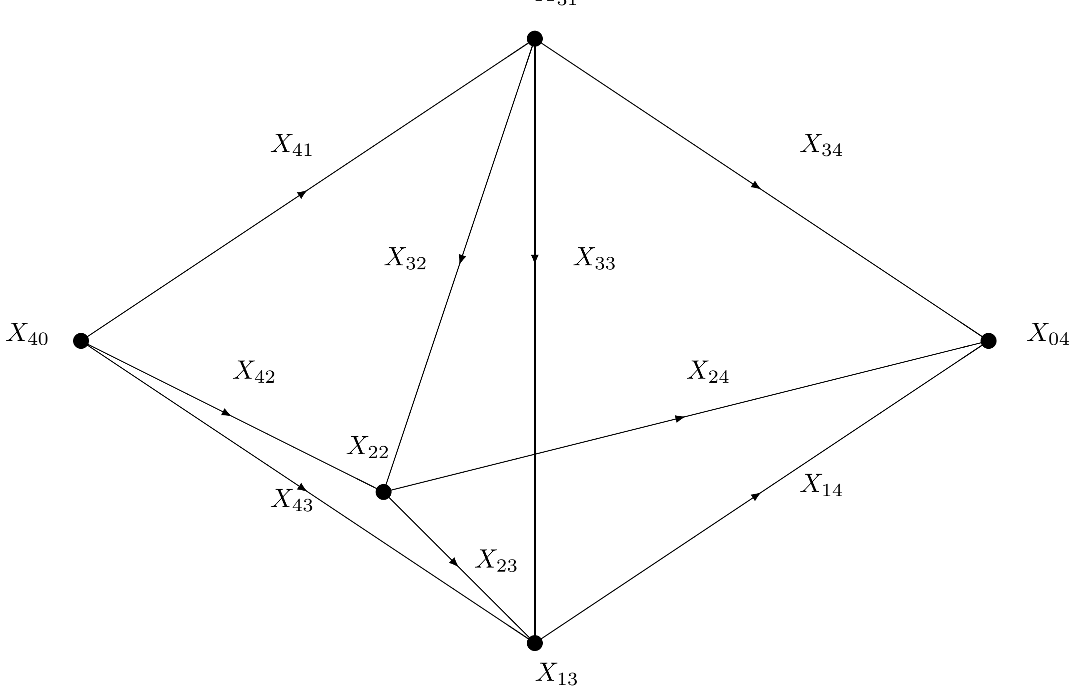

Fig. 2

4. $\mathbb{C}^*$-actions on Grassmann bundles

This section defines in the compact Kaehler case the Grassmann Graph construction of [1, pp. 120-121].

Let E, F be vector bundles of ranks k and n, respectively, on an analytic space M. Let $G(k, E\oplus F) \to M$ denote the Grassmann bundle whose fibre at each $x \in M$ is $G(k, E_x \oplus F_x)$, the Grassmannian of k-planes in $E_x \oplus F_x$. Define a $\mathbb{C}^*$-action on $G(k, E \oplus F)$ as the fibrewise $\mathbb{C}^*$-action. Let

$$\pi_1 : E \oplus F \longrightarrow E$$
$$\pi_2 : E \oplus F \longrightarrow F$$

and

$$\pi : G(k, E \oplus F) \longrightarrow M$$

be the projections. Any $p \in G(k, E \oplus F)$ is represented by a k-plane H in $E_x \oplus F_x$ where $x = \pi(x)$. $\pi_1(H)$ and $\pi_2(H)$ are linear subspaces of E_x and F_x, respectively. The total space $G(k, E \oplus F)$ can be decomposed into $\mathbb{C}^*$-equivariant subbundles

$$X_{ij} = \big\{ [H] \in G(k, E \oplus F) \ \big| \ \dim \pi_1(H) = i,\ \dim \pi_2(H) = j \ \big\}$$

where $k - \min(k, n) \le i \le k$, $\ 0 \le j \le \min(k, n)$, and $i + j \ge k$. It is easy to see that

$$X_{ij} \cong G(i, E) \times G(j, F) \ \text{ if } \ i + j = k,$$

which are the fixed point sets of the $\mathbb{C}^*$-action. Let

$$\mathrm{Hom}(E, F) \longrightarrow M$$

be the bundle of morphisms from E to F and let

$$j : \mathrm{Hom}(E, F) \longrightarrow G(k, E \oplus F)$$

be the natural inclusion defined fibrewise as

$$j_x(\Phi) = \mathrm{graph}(\Phi \mid E_x) = \big\{ \big(e, \Phi(e)\big) \in E_x \oplus F_x \big\}.$$

Recall that $\mathbb{C}$ can be imbedded into $\mathbb{P}^1$ as

$$\mathbb{C} \longrightarrow \mathbb{P}^1$$
$$\lambda \longrightarrow [1 : \lambda],$$

[1, p. 120]. Define a $\mathbb{C}^*$-action on $G(k, E \oplus F) \times \mathbb{P}^1$

$$\mathbb{C}^* \times G(k, E \oplus F) \times \mathbb{P}^1 \longrightarrow G(k, E \oplus F) \times \mathbb{P}^1$$

as

$$\big(\lambda, p, [\lambda_0 : \lambda_1]\big) \longrightarrow \big(\lambda \cdot p, [\lambda_0 : \lambda\lambda_1]\big)$$

where $\lambda \cdot p$ is the $\mathbb{C}^*$-action which is defined above. Also define the $\mathbb{C}^*$-action on $M \times \mathbb{C}$,

$$\mathbb{C}^* \times M \times \mathbb{C} \longrightarrow M \times \mathbb{C}$$

as

$$\big(\lambda, x, t\big) \longrightarrow \big(x, \lambda t\big).$$

Every $\Phi \in \mathrm{Hom}(E,F)$ defines an equivariant imbedding $\overline{s}(\Phi)$ of $M\times\mathbb{C}$ into $G(k, E\oplus F)\times\mathbb{P}^1$,

$$\overline{s}(\Phi) : M \times \mathbb{C} \longrightarrow G(k, E\oplus F) \times \mathbb{P}^1$$

where

$$\overline{s}(\Phi)(x,\lambda) = \big(\big[j_x(\lambda\Phi_x)\big], [1:\lambda]\big).$$

Let $s(\Phi) = \mathrm{pr}(\overline{s}(\Phi))$ where pr is the projection

$$\mathrm{pr} : G(k, E\oplus F)\times\mathbb{P}^1 \longrightarrow G(k, E\oplus F).$$

$s(\Phi)(M,\lambda)$ is the graph of $\lambda\Phi$. Now define

$$\mathbf{Z}_\infty = \lim_{\lambda\to\infty} s(\Phi)(M,\lambda).$$

THEOREM 1. *If M is a compact Kaehler manifold, then for any $\Phi \in \mathrm{Hom}(E,F)$ the corresponding $\mathbf{Z}_\infty$ is an analytic cycle.*

PROOF: Let $\rho : \mathbb{C}^*\times G(k, E\oplus F) \to G(k, E\oplus F)$ be the $\mathbb{C}^*$-action defined above. Consider M as a subspace of $G(k, E\oplus F)$ by the imbedding $s(\Phi)(M,\lambda)$; i.e. identify M and the graph of Φ. define a holomorphic map

$$A : M\times\mathbb{C}^* \longrightarrow G(k, E\oplus F)$$

as

$$A(m,t) = s(\Phi)(m,t),$$

where $m\in M$ and $t\in\mathbb{C}^*$. This map is equivariant with respect to ρ and the trivial action of $\mathbb{C}^*$ on $M\times\mathbb{C}^*$, multiplication in the second component; for if $\lambda\in\mathbb{C}^*$ then

$$\begin{aligned} A(m,\lambda\cdot t) &= s(\Phi)(m,\lambda t) \\ &= s(\lambda\Phi)(m,t) \\ &= \lambda\cdot s(\Phi)(m,t) \\ &= \rho(\lambda, s(\Phi)(m,t)) \\ &= \rho(\lambda, A(m,t)) \end{aligned}$$

hence equivariance. But Sommese has shown that if $\psi : Y\times\mathbb{C}^* \to X$ is a holomorphic map equivariant with respect to the trivial action of $\mathbb{C}^*$ on $Y\times\mathbb{C}^*$ and the action of $\mathbb{C}^*$ on X with fixed points then ψ extends meromorphically to $Y\times\mathbb{P}^1$, [8,p. 111 (Lemma II-B)]. Thus A extends meromorphically to

$$A' : M\times\mathbb{P}^1 \longrightarrow G(k, E\oplus F).$$

Let T be the closure of the graph of A in $M\times\mathbb{P}^1\times G(k, E\oplus F)$.

By the definition of a meromorphic map, T is an analytic space. Since

$$M\times\{\infty\}\times\mathbf{Z}_\infty = T\cap\big(M\times\{\infty\}\times G(k, E\oplus F)\big),$$

being the intersection of two analytic spaces it is analytic. If pr: $M\times\{\infty\}\times\mathbf{Z}_\infty \to M$ is the projection, then for any $m\in M$, $\mathrm{pr}^*(M) = \{m\}\times\mathbf{Z}_\infty$ is an analytic cycle, from which it follows that $\mathbf{Z}_\infty$ is analytic as desired.

$\mathbf{Z}_\infty$ is called the cycle at infinity corresponding to the map Φ. Notice that there is an alternate definition of $\mathbf{Z}_\infty$ see [1, p. 121];

Let W be the closure of $\overline{s}(\Phi)(M \times \mathbb{C})$ in $G(k, E \oplus F) \times \mathbb{P}^1$. Then $\mathbf{Z}_\infty \times \{\infty\}$ is the intersection of W and $G(k, E \oplus F) \times \{\infty\}$.

In the algebraic category W is an algebraic variety but in the analytic category the observation that W can be obtained through a $\mathbb{C}^*$-action with fixed points on a compact Kaehler manifold is crucial in concluding that it is analytic.

Clearly $\{Z_\lambda = s(\Phi)(M, \lambda)\}$ defines a family of cycles which are algebraically and hence homologically equivalent.

5. Graphs of complexes

In this section we define the Grassmann Graph construction and the cycle at infinity associated to a complex of vector bundles. This construction was first introduced by MacPherson and used by Baum, Fulton and MacPherson to prove Riemann-Roch theorem for singular algebraic varieties, [1] and [6].

Consider a complex of vector bundles on M,

$$(E.): \quad 0 \longrightarrow E_m \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0.$$

Denote the maps by γ_i, i.e.

$$\gamma_i : E_i \longrightarrow E_{i-1}$$

where $i = 0, ..., m$, $E_{-1} = 0$.

Assume that there is a subvariety S of M such that $(E.)$ is exact on $M - S$.

Let

$$G_I = G(\mathrm{rank} E_i,\ E_i \oplus E_{i-1}\ ), \quad i = 1, ..., m.$$

and let

$$\tau_i \longrightarrow G_i \quad \text{the tautological bundle,} \quad i = 1, ..., m.$$

Define

$$G = G_0 \times_M \cdots \times_M G_m$$

where $\times_M$ denotes the bundle product on M. On G let τ_i denote the pull back of $\tau_i \to G_i$ by the projection $\mathrm{pr}_i : G \to G_i$ of the i-th component, $i = 0, ..., m$.

Let

$$\tau = \tau_0 - \tau_1 + \cdots + (-1)^m \tau_m$$

be the virtual tautological bundle on G. Recalling the definition of s from the previous section, for any $\lambda \in \mathbb{C}$ define an imbedding

$$s^i_\lambda : M \longrightarrow G_i$$

as

$$s^i_\lambda(x) = s(\gamma_i)(x, \lambda)$$

where $i = 0, ..., m$. Then define for any $\lambda \in \mathbb{C}$ an imbedding

$$s_\lambda : M \longrightarrow G$$

as

$$s_\lambda(x) = \big(s_\lambda^0(x), ..., s_\lambda^m(x)\big).$$

Using $s_\lambda(M)$ we define

$$\mathbf{Z}_\infty = \lim_{\lambda \to \infty} s_\lambda(M)$$

to be the cycle at infinity corresponding to the complex $(E.)$.

Let $\pi : G \to M$ be the natural projection. Recalling that S is the set off which $(E.)$ is exact we have the following result: (For proofs see [1, p. 121].)

Theorem (Baum, Fulton, MacPherson). *The cycle $\mathbf{Z}_\infty$ has a unique decomposition $\mathbf{Z}_\infty = \mathbf{Z}_* + \mathbf{M}_*$, where*
1) π maps M meromorphically onto M.
2) $\pi : \mathbf{M}_* - \pi^{-1}(S) \longrightarrow M - S$ is a biholomorphism.
3) π maps Z into S.
4) τ restricts on $\mathbf{M}_*$ to the zero bundle.

Remark. By Theorem 1 of the previous section, $\mathbf{Z}_\infty$ is a product of analytic cycles in the product bundle G, hence this theorem can be stated in the analytic category as above. Any cycle can be written as a sum of irreducible cycles. the decomposition of $\mathbf{Z}_\infty$ is such a sum. For a proof of (4) see [1, p. 122].

Finally we define two residues on S. Let E be the virtual bundle $E_0 - E_1 + \cdots + (-1)^m E_m$ on M. Then $\tau \,\big|\, Z_0$ is isomorphic to E since $Z_0 \cong M$. Since $Z_)$ and $\mathbf{Z}_\infty$ are rationally equivalent

$$c(E) \cap [M] = c(\tau) \cap Z_0 = c(\tau) \cap \mathbf{Z}_\infty$$

where $c(\cdot)$ denotes the Chern class and $\cap$ denotes the cap product. Since $\mathbf{Z}_\infty$ decomposes

$$\begin{aligned} c_i(\tau) \cap \mathbf{Z}_\infty &= c_i(\tau) \cap (\mathbf{Z}_* + \mathbf{M}_*) \\ &= c_i(\tau) \cap \mathbf{Z}_* + c_i(\tau) \cap \mathbf{M}_* \\ &= c_i(\tau) \cap \mathbf{Z}_* \end{aligned}$$

where $i > 0$ and the last equality follows since $\tau \,\big|\, \mathbf{M}_* = 0$ by (4) of the above theorem.

Define

$$c_S^i(E.) = \pi_*\big(c_i(\tau) \cap \mathbf{Z}_*\big) \in H_*(S : \mathbb{C}).$$

Similarly let $ch(\cdot)$ denote the Chern character, then

$$\begin{aligned} ch(E) \cap [M] &= ch(\tau) \cap Z_0 \\ &= ch(\tau) \cap \mathbf{Z}_\infty \\ &= ch(\tau) \cap \mathbf{Z}_* + ch(\tau) \cap \mathbf{M}_* \\ &= ch(\tau) \cap \mathbf{Z}_*. \end{aligned}$$

Similarly define

$$ch_S(E.) = \pi_*\big(ch(\tau) \cap \mathbf{Z}_*\big) \in H_*(S; \mathbb{C}).$$

For basic properties of $ch(E.)$ in the algebraic category see [1, pp. 121-126]. We will use $c^i_S(E)$ for calculating the Baum-Bott residue of singular holomorphic foliations in [7].

6. Examples

1) Let E, F be vector bundles on M and $\psi \in \mathrm{Hom}(E, F)$.Then the graph $\Gamma(\psi)$ of ψ gives rise to a cycle at infinity $\mathbf{Z}_\infty$. Let rank$E = k$, rank$F = n$, and $m = \min\{k, n\}$. For $i = 0, 1, ..., m$, let $B_i = X_{k-ii}$, where X_{ij} is as defined in Section 4. $B_0, ..., B_m$ are the components of the fixed point set B under the $\mathbb{C}^*$-action on the Grassmann bundle $G(k, E \oplus F)$. To understand the structure of $\mathbf{Z}_\infty$ we describe its intersection with B. For this purpose define the following sets

$$\Sigma_i = \{p \in M \mid \text{ rank } \psi_p \leq i \}, \;\; i = 0, ..., r$$

where r is the generic rank of ψ. the behaviour of $\mathbf{Z}_\infty$ can now be described as follows:

$$\big(\mathbf{Z}_\infty \cap B_i\big)_p \neq \emptyset \quad \text{iff} \quad p \in \Sigma_t \;\text{ and }\; t \geq i \geq r.$$

2) We want to show that the Hironaka Blow-up at a point can be recovered as a Grassmann Graph construction. The problem is local so let M be an open set in $\mathbb{C}^n$. Define two trivial bundles L and F as

$$L = M \times \mathbb{C} \;\text{ and }\; F = M \times \mathbb{C}^n.$$

Define a morphism $\theta \in \mathrm{Hom}(L, F)$ as:

$$\theta(p, t) = (p, tp) \;\text{ for }\; p \in \mathbb{C}^n, \; t \in \mathbb{C}.$$

The cycle at infinity $\mathbf{Z}_\infty$ corresponding to θ intersects the sink of $G(1, L \oplus F)$ in $\mathbf{M}_*$, that is $\mathbf{Z}_\infty = \mathbf{M}_* + \mathbf{Z}_*$. $\mathbf{M}_*$is the Hironaka Blow-up of M at the origin. We can see this as follows. Let $p = (p_1, ..., p_n) \in M = \mathbb{C}^n$. We also identify $\mathbb{P}(L \oplus F)$ with $\mathbb{P}^n$. There is a $\mathbb{C}^*$-action

$$\mathbb{C}^* \times M \times \mathbb{P}^n \longrightarrow M \times \mathbb{P}^n$$

given as

$$\big(\lambda, p, [y_0 : y_1 : \cdots : y_n]\big) \to \big(p, [y_0 : \lambda y_1 : \cdots : \lambda y_n]\big).$$

The graph of θ has the form

$$\Gamma(\theta) = \big\{(p, [1 : p_1 : \cdots : p_n]) \in M \times \mathbb{P}^n\big\}.$$

The $\mathbb{C}^*$-action moves $\Gamma(\theta)$ as

$$\lambda \cdot \Gamma(\theta) = \big\{(p, [1 : \lambda p_1 : \cdots : \lambda p_n]) \in M \times \mathbb{P}^n\big\}.$$

Consider the usual imbedding of $\mathbb{C}^*$ in $\mathbb{P}^1$ as $\lambda = [1 : \lambda] = [\lambda_0 : \lambda_1]$, where $\lambda = \lambda_1/\lambda_0$. Since $\lambda \to \infty$ iff $\lambda_0 \to 0$ with $\lambda_1 \neq 0$, we have the following limit

$$\begin{aligned}
\mathbf{Z}_\infty &= \lim_{\lambda \to \infty} \lambda \cdot \Gamma(\theta) \\
&= \lim_{\lambda_0 \to 0} \big\{(p, [\lambda_0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]) \in M \times \mathbb{P}^n\big\} \\
&= \big\{(p, [0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]) \in M \times \mathbb{P}^n\big\}.
\end{aligned}$$

Clearly $(p, [0 : \lambda_1 p_1 : \cdots : \lambda_1 p_n]) \in M \times \mathbb{P}^n$ can be considered as a point $(p, [x_1 : \cdots : x_n])$ in $M \times \mathbb{P}^{n-1}$ such that

$$p_j x_i = p_i x_j, \;\; i \neq j, \;\; 1 \leq i, j \leq n.$$

From here it is easy to see that the intersection of $\mathbf{Z}_\infty$ with the sink of the $\mathbb{C}^*$-action is the Hironaka blow-up of M at the origin.

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DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06533 ANKARA, TURKEY

UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., CANADA
E-mail address: `sertoz@@fen.bilkent.edu.tr`