

Ec551 Studies in Growth and General Equilibrium
Midterm I
ANSWERS

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This is an open book, take home exam. You have 24 hours to complete your answers.

I. SHORT Questions: (20 points each)

1) Consider the following two growth models: In the first, there is standard Ramsey-Cass-Koopmans model of exogenous growth with exogenous labor-augmenting technological progress at the rate γ . The second economy experiences endogenous growth via the Rebello-type of convex technology, with the following capital accumulation equation in the aggregate:

$$\dot{K} = AK - C - \delta K$$

Contrast the effect of growth in the population at the constant rate n . Explain intuitively any differences in the growth rate in per capita incomes across the two models.

ANSWER

In the standard Solow-Ramsey- Coopmans neoclassical model steady state per capita growth rate is always equal to the rate of growth of per capita productivity, independent of the labor force growth rate. Thus, with exogenous technological change per capita growth rate will be γ . Countries with higher population will necessarily have lower ratios of per capita income, but it will grow at the same rate of γ .

In contrast, in the AK model, capital accumulation equation in per capita terms is:

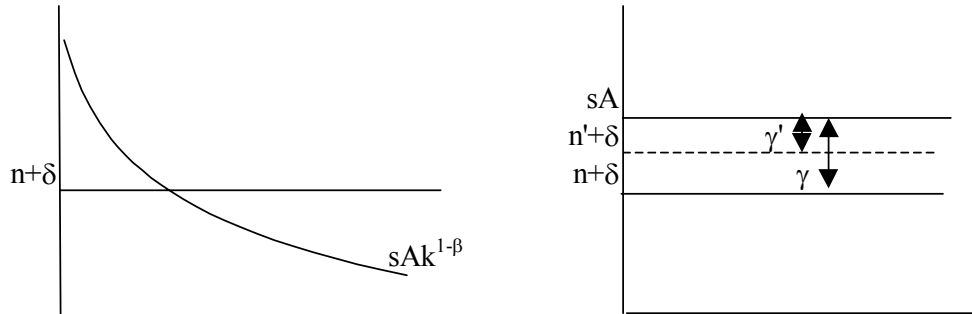
$$\dot{k} = (A - \delta - n)k - c$$

which with the Euler equation on consumption, $\frac{\dot{c}}{c} = \frac{1}{\sigma} [f'(k) - \rho]$ yields the following accumulation equation over time:

$$\dot{k} = (a - \delta - n)k - c(0)e^{\frac{1}{\sigma}[A - \delta - n - \rho]t}$$

Observe that the model has no transitional dynamics and the consumption growth rate doesnot depend on capital per worker.

In this model, an increase in the population growth rate acts like a decline in the productivity parameter, A, with permanently reducing the rate of per capita growth.



2) Consider the following model of endogenous growth due to Arrow (1962). The technology facing each firm is given by

$$y = k^\alpha \kappa^\beta$$

where $\kappa = L \sum_i k_i$ and $\alpha + \beta = 1$. Thus, there is an externality on the part of each firm originating by the aggregate size of the capital stock, κ . Suppose that there is no capital depreciation and the budget constraint is:

$$\dot{k} = k^\alpha \kappa^\beta - c$$

Consumers preferences are given by the intertemporal felicity,

$$U = \int_0^{\infty} u(c_t) e^{-\rho t} dt$$

where $u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$, with $\sigma > 0$, and $\rho > 0$ is the rate of time preference.

Find the steady state growth rate in this economy. Show that this rate depends on the size of the labor supply, L , and that it generates scale effects in growth; *i.e.* economies which have high levels of labor stocks tend to grow faster in the context of this model. Comment on the “desirability” of this result from the point of realistic empirics of growth.

ANSWER

The Hamiltonian for the Consumer’s problem is:

$$H = e^{-\rho t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] + v \left[k^\alpha \kappa^\beta - c \right]$$

$$\frac{\partial H}{\partial c} = e^{-\rho t} c^{-\sigma} - v = 0$$

$$\frac{\partial H}{\partial k} = v \alpha k^{\alpha-1} \kappa^\beta = -\dot{v}$$

whose solution yields the steady state growth equation:

$$\begin{aligned} \frac{\dot{c}}{c} = \gamma &= \frac{1}{\sigma} \left[\alpha k^{-(1-\alpha)} \kappa^\beta - \rho \right] \\ &= \frac{1}{\sigma} \left[\alpha k^{-(1-\alpha)} (kL)^\beta - \rho \right] \\ &= \frac{1}{\sigma} \left[\alpha k^{-(1-\alpha-\beta)} L^\beta - \rho \right] \end{aligned}$$

Thus, in this economy there are scale effects on growth; *i.e.*, the steady state rate of growth increases with increases in population, L . The model predicts that economies with higher population will tend to grow faster, which is hard to justify empirically.

I. LONG Questions: (30 points each)

1) Consider the Ramsey-Cass-Koopmans problem of exogenous growth where in a given economy, single consumption good, y , is produced by a competitive firm with the neo-classical production technology with *constant* returns to *scale*, but *diminishing* returns to both *factors*:

$$y = f(k)$$

where k is capital per worker and $f(\cdot)$ is a continuous function representing the underlying technology.

The representative consumer maximizes,

$$U = \int_0^{\infty} u(c_t) e^{-\rho t} dt$$

where $u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$, with $\sigma > 0$, and $\rho > 0$ is the rate of time preference. Assume for simplicity that the population growth rate in the economy is zero. The consumer's intertemporal budget constraint satisfies that asset accumulation equals wage and interest income net of consumption. Thus,

$$\dot{a} = w + ra - c$$

where a stands for assets, w is wage income, r is interest rate on asset holdings, c is per capita consumption.

Law of motion of capital per labor obeys:

$$\dot{k} = f(k) - c - \delta k$$

where δ is the rate of depreciation.

- a) Assuming that the economy is closed to foreign trade, solve the consumer's and the producer's intertemporal optimization problems.
- b) Now suppose that the government buys output at the rate $g(t)$ and its purchases are assumed not to affect utility from private consumption, nor future output. They are devoted to public consumption with no utility implications, and they are financed by lump sum taxes. Thus the government runs a balanced budget. Suppose that the government purchases are unexpected and are of a permanent nature. How will the long run equilibrium of the economy be affected? Show in a phase diagram.

- c) Now suppose that the government switches to a policy of taxing income at rate t . The proceeds $g(t)$ now are rebated lump sum to the consumer. Again assuming that the regime change is thoroughly unanticipated and permanent, find the effect of this policy on the long run equilibrium of the economy and show with the aid of a phase diagram.

ANSWER

The consumer's Hamiltonian is:

$$H = \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + v[(1-t_y)f(k) - t_L - c - \delta k]$$

First order conditions :

$$\frac{\partial H}{\partial c} = c^{-\sigma} e^{-\rho t} - v = 0$$

$$\frac{\partial H}{\partial k} = v[(1-t_y)f'(k) - \delta] = -\dot{v}$$

Then we obtain the Euler equation for this system :

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} [(1-t_y)f'(k) - \delta - \rho]$$

Where I have denoted t_L for lump sum taxes, and t_y for the income tax rate. The *Euler Equation* indicates that if the after tax net rate of return on capital (the first two terms in the brackets) is equal to the subjective rate of time preference, ρ , then consumption does not change over time. If the net rate of return is larger than ρ , then individuals postpone their consumption to the future, thus $\dot{c}/c > 0$.

- b) With the introduction of government spending financed by taxes on private agent, the nature of transitional dynamics change.

The transitional dynamics are governed by the following two laws of motion:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} [f'(k) - \delta - \rho]$$

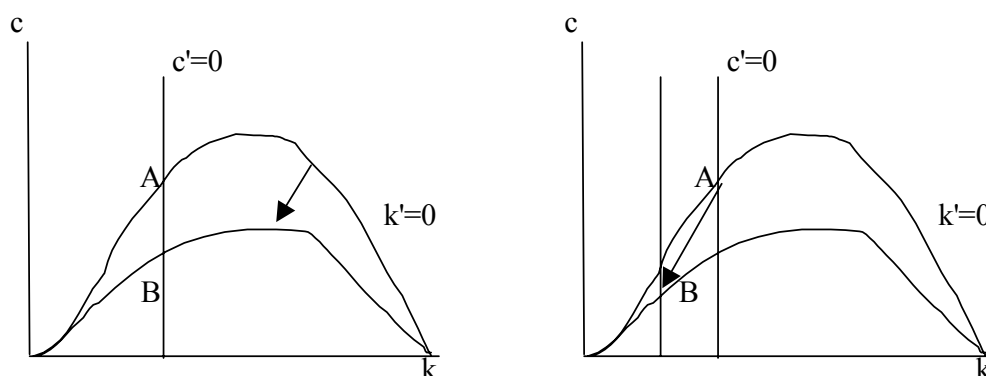
$$\dot{k} = (1-t_y)f(k) - t_L - c - \delta k$$

At the steady state, $dc/dt = dk/dt = 0$; thus

$$(1 - t_y)f'(k) = \delta + \rho$$

and

$$c = (1 - t_y)f(k) - t_L - \delta k$$



In the figures the original steady state of the economy is given at point A. Now, suppose that t_L is changed with no change in income taxes. This policy leaves the $c'=0$ schedule unchanged, but shifts the $k'=0$ schedule downwards. The new steady state is point B in left hand panel, with steady state capital being unchanged, but per capita consumption being lowered.

c) Now suppose that the income tax rate is increased. From the transition equations, it can be seen that consumption equation is affected and $c'=0$ schedule shifts leftwards. Also the net after tax rate of return is lowered, steady state k is lowered. The income tax rate also lowers the capital accumulation schedule inwards. The new steady state is obtained at point B in right hand panel.

So the difference in the effects of t_Y and t_L is that while both taxes lower per capita consumption in the steady state, the lump sum tax rate leaves the capital per labor unaffected. The t_Y decreases the capital per labor permanently as it reduces the net rate of return to capital accumulation.

2) Suppose that in a given economy, single consumption good, y , is produced by a competitive firm with the production technology:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

where K is aggregate capital and L is aggregate labor. The term A is the level of productivity (efficiency units of labor) with its initial level normalized at $A_0=1$.

The representative consumer maximizes,

$$U = \int_0^{\infty} u(c_t) e^{-\rho t} dt$$

where $u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$, with $\sigma > 0$, and $\rho > 0$ is the rate of time preference.

Law of motion of capital obeys

$$\dot{K}_t = Y_t - C_t - \delta K_t$$

and the productivity coefficient grows at the exogenous rate: $\frac{\dot{A}_t}{A_t} = \gamma$.

- Derive the steady-state consumption rule (the Euler equation) which maximizes the consumer's intertemporal preferences.
- From the equation derived in (a) above, obtain the steady state level of capital stock in terms of γ , ρ , σ , and δ .
- Now define the steady-state saving rate in this economy as:

$$s_t = \frac{Y_t - C_t}{Y_t}$$

Show that the state saving rate is given by: $s^* = (\gamma + \delta)k^{1-\alpha}$.

- Use your answers in (b) and (c) to show that steady state rate of savings is positively related to γ , only if $\sigma < 1 + \frac{\rho}{\delta}$.

Empirical work suggests a subjective discount rate of 0.05. Depreciation rate can be regarded at around 0.10. Find the range of values of σ for which $\partial s / \partial \gamma$ is positive.

ANSWER

a) Observe that the law of motion of capital can be expressed in terms of efficiency units as follows:

Since,

$$\begin{aligned}\frac{dK}{dt} &= \frac{dk_t A_t}{dt} \\ &= \frac{dk_t}{dt} A_t + k_t \frac{dA_t}{dt} \\ &= (\dot{k}_t + k_t \gamma) A_t\end{aligned}$$

$$\dot{K} = \frac{dK}{dt} = Y_t - C_t - \delta K_t \quad \text{implies :}$$

$$(\dot{k}_t + k_t \gamma) A_t = Y_t - C_t - \delta K_t$$

$$\dot{k}_t = y_t - \frac{C_t}{A_t} - (\delta + \gamma) k_t$$

where $k_t = K_t/A_t$, and $y_t = Y_t/A_t$; with $\frac{\dot{A}_t}{A_t} = \gamma$.

Observe in equilibrium, $a=k$. Construct the Hamiltonian with $y_t = k_t^\alpha$.

$$H = e^{-\rho t} \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} \right] + v \left[k^\alpha - \frac{C_t}{A_t} - (\delta + \gamma) k_t \right]$$

$$\frac{\partial H}{\partial c} = e^{-\rho t} c^{-\sigma} - v = 0$$

$$\frac{\partial H}{\partial k} = \left[\alpha k^{\alpha-1} - (\delta + \gamma) \right] v = -\dot{v}$$

Equating the first order conditions we obtain :

$$\rho + \sigma \frac{\dot{C}}{C} - \frac{\dot{A}}{A} = -\frac{\dot{v}}{v} = \alpha k_t^{\alpha-1} - \delta - \gamma$$

From where, the steady state value of capital per labor can be obtained:

$$k^* = \left[\frac{\sigma\gamma + \delta + \rho}{\alpha} \right]^{\frac{1}{\alpha-1}}$$

b) To obtain the steady state saving rate:

$$s_t = \frac{Y_t - C_t}{Y_t}$$

since :

$$\dot{k}_t = y_t - \frac{C_t}{A_t} - (\delta + \gamma)k_t$$

at steady state, $\dot{k} = 0$ implies:

$$\frac{C}{A} = y - (\delta + \gamma)k$$

Thus,

$$\begin{aligned} s &= \frac{y - C/A}{y} \\ &= \frac{(\delta + \gamma)k}{k^\alpha} \\ &= (\delta + \gamma)k^{1-\alpha} \end{aligned}$$

c) Using the steady state value of k^*

$$s = (\delta + \gamma) \left[\frac{\sigma\gamma + \delta + \rho}{\alpha} \right]^{-1}$$

The derivative of the steady state saving rate with respect to γ gives:

$$\frac{\partial s}{\partial \gamma} = -(\delta + \gamma) \left(\frac{\sigma}{\alpha} \right) \left[\frac{\sigma\gamma + \rho + \delta}{\alpha} \right]^{-2} + \left[\frac{\sigma\gamma + \rho + \delta}{\alpha} \right]^{-1}$$

This term will be positive if:

$$-(\delta + \gamma) \left(\frac{\sigma}{\alpha} \right) \left[\frac{\sigma\gamma + \rho + \delta}{\alpha} \right] > 0$$

$$\sigma\gamma + \rho + \delta > (\delta + \gamma)\sigma$$

$$\rho + \delta > \sigma\delta$$

$$1 + \frac{\rho}{\delta} > \sigma$$

Using $\rho=0.05$, $\delta=0.10$, we observe that $\partial s/\partial\gamma$ is positive for $\sigma < 1.5$.